(%i1)

```
/* define special summation function */
f(i,j) := sum(R[i,j,sigma,0]*gContr[i,sigma]*gContr[j,0],sigma,0,3)
        + sum(R[i,j,sigma,1]*gContr[i,sigma]*gContr[j,1],sigma,0,3)
        + sum(R[i,j,sigma,2]*gContr[i,sigma]*gContr[j,2],sigma,0,3)
        + sum(R[i,j,sigma,3]*gContr[i,sigma]*gContr[j,3],sigma,0,3);
```

(%o1) $f(i,j) := sum(R_{i,j,\sigma,0}\, gContr_{i,\sigma}\, gContr_{j,0}, \sigma, 0, 3) +$

$sum(R_{i,j,\sigma,1}\, gContr_{i,\sigma}\, gContr_{j,1}, \sigma, 0, 3) +$

$sum(R_{i,j,\sigma,2}\, gContr_{i,\sigma}\, gContr_{j,2}, \sigma, 0, 3) +$

$sum(R_{i,j,\sigma,3}\, gContr_{i,\sigma}\, gContr_{j,3}, \sigma, 0, 3)$

(%i2)

```
/* define coordinate vector */
array(x, 3);
[x[0],x[1],x[2],x[3]]: [t, r, theta, phi];
```

(%o2) x

(%o3) $[t, r, \theta, \varphi]$

(%i4)

```
/* define coordinate dependent functions */
depends([y], [r]);
```

(%o4) $[y(r)]$

(%i5)

```
y: 2*G*M/(r*c^2)$
```

(%i6)

```
/* g1 is symm. metric with indices 1...4 */
g1: matrix(
  [-(1-y),0,0,0],
  [0,(1-y)^-1,0,0],
  [0,0,r^2,0],
  [0,0,0,r^2*sin(theta)^2]
);
```

(%o6)
$$\begin{bmatrix} \frac{2\,G\,M}{c^2\,r}-1 & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2\,G\,M}{c^2\,r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\,sin(\theta)^2 \end{bmatrix}$$

(%i7)

```
/* contravariant g is inverse of g */
gContr1: ratsimp(invert(g1));
```

(%o7)
$$\begin{bmatrix} \frac{c^2 r}{2GM - c^2 r} & 0 & 0 & 0 \\ 0 & -\frac{2GM - c^2 r}{c^2 r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin(\theta)^2} \end{bmatrix}$$

(%i8)

```
/* g1 and gContr1 are transformed to g and gContr (indices 0...3) */
for mu:0 thru 3 do {
for nu:0 thru 3 do {
    g      [mu,nu]: g1      [mu+1, nu+1],
    gContr[mu,nu]: gContr1[mu+1, nu+1]
}}$
```

(%i9)

```
/* computation of Christoffel symbols Gamma^sigma_mu_nu */
for sigma:0 thru 3 do {
for mu:0 thru 3 do {
for nu:0 thru 3 do {
    Gamma[sigma,mu,nu]  :
    /* rho sum by function call: */
    sum(
        1/2 * gContr[sigma,rho]*(
        diff(g[nu,rho],x[mu] ) +
        diff(g[rho,mu],x[nu] ) -
        diff(g[mu,nu] ,x[rho])),
    rho, 0, 3),
    /* evaluate differentiation dy/dr */
    Gamma[sigma,mu,nu]: ev(Gamma[sigma,mu,nu],diff)
}}}$
```

(%i10)

```
/* dislplay Gamma's being different from zero */
for i:0 thru 3 do {
for j:0 thru 3 do {
for k:0 thru 3 do {
    if Gamma[i,j,k] # 0 then {
        display(Gamma[i,j,k])
    }}}}$
```

$$\Gamma_{0,0,1} = -\frac{GM}{r(2GM - c^2 r)}$$

$$\Gamma_{0,1,0} = -\frac{GM}{r(2GM - c^2 r)}$$

$$\Gamma_{1,0,0} = -\frac{GM(2GM - c^2 r)}{c^4 r^3}$$

$$\Gamma_{1,1,1} = \frac{G\,M\,(2\,G\,M - c^2\,r)}{c^4\,r^3\left(1 - \frac{2\,G\,M}{c^2\,r}\right)^2}$$

$$\Gamma_{1,2,2} = \frac{2\,G\,M - c^2\,r}{c^2}$$

$$\Gamma_{1,3,3} = \frac{\sin(\theta)^2\,(2\,G\,M - c^2\,r)}{c^2}$$

$$\Gamma_{2,1,2} = \frac{1}{r}$$

$$\Gamma_{2,2,1} = \frac{1}{r}$$

$$\Gamma_{2,3,3} = -\cos(\theta)\,\sin(\theta)$$

$$\Gamma_{3,1,3} = \frac{1}{r}$$

$$\Gamma_{3,2,3} = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\Gamma_{3,3,1} = \frac{1}{r}$$

$$\Gamma_{3,3,2} = \frac{\cos(\theta)}{\sin(\theta)}$$

```
(%i11) /* compute Riemann tensor elements */
       for rho:0 thru 3 do {
       for sigma:0 thru 3 do {
       for mu:0 thru 3 do {
       for nu:0 thru 3 do {
           R[rho,sigma,mu,nu] :
           diff(Gamma[rho,nu,sigma],x[mu]) -
           diff(Gamma[rho,mu,sigma],x[nu]) +
           /* lambda sums by function call: */
           sum(
               Gamma[rho,mu,lambda] * Gamma[lambda,nu,sigma] -
               Gamma[rho,nu,lambda] * Gamma[lambda,mu,sigma],
               lambda, 0, 3)
       }}}}$
```

```
(%i12) /* dislplay R's being different from zero */
       for i:0 thru 3 do {
       for j:0 thru 3 do {
       for k:0 thru 3 do {
       for l:0 thru 3 do {
           R[i,j,k,l] : /*ratsimp*/(factor(R[i,j,k,l])),
           if R[i,j,k,l] # 0 then display(R[i,j,k,l])
           }}}}$
```

$$R_{0,1,0,1} = -\frac{2\,G\,M}{r^2\,(2\,G\,M - c^2\,r)}$$

$$R_{0,1,1,0} = \frac{2\,G\,M}{r^2\,(2\,G\,M - c^2\,r)}$$

$$R_{0,2,0,2} = -\frac{G\,M}{c^2\,r}$$

$$R_{0,2,2,0} = \frac{G\,M}{c^2\,r}$$

$$R_{0,3,0,3} = -\frac{\sin(\theta)^2\,G\,M}{c^2\,r}$$

$$R_{0,3,3,0} = \frac{\sin(\theta)^2\,G\,M}{c^2\,r}$$

$$R_{1,0,0,1} = -\frac{2\,G\,M\,(2\,G\,M - c^2\,r)}{c^4\,r^4}$$

$$R_{1,0,1,0} = \frac{2\,G\,M\,(2\,G\,M - c^2\,r)}{c^4\,r^4}$$

$$R_{1,2,1,2} = -\frac{G\,M}{c^2\,r}$$

$$R_{1,2,2,1} = \frac{G\,M}{c^2\,r}$$

$$R_{1,3,1,3} = -\frac{\sin(\theta)^2\,G\,M}{c^2\,r}$$

$$R_{1,3,3,1} = \frac{\sin(\theta)^2\,G\,M}{c^2\,r}$$

$$R_{2,0,0,2} = \frac{G\,M\,(2\,G\,M - c^2\,r)}{c^4\,r^4}$$

$$R_{2,0,2,0} = -\frac{G\,M\,(2\,G\,M - c^2\,r)}{c^4\,r^4}$$

$$R_{2,1,1,2} = -\frac{G\,M}{r^2\,(2\,G\,M - c^2\,r)}$$

$$R_{2,1,2,1} = \frac{G\,M}{r^2\,(2\,G\,M - c^2\,r)}$$

$$R_{2,3,2,3} = \frac{2\,\sin(\theta)^2\,G\,M}{c^2\,r}$$

$$R_{2,3,3,2} = -\frac{2\,\sin(\theta)^2\,G\,M}{c^2\,r}$$

$$R_{3,0,0,3} = \frac{G\,M\,(2\,G\,M - c^2\,r)}{c^4\,r^4}$$

$$R_{3,0,3,0} = -\frac{G\,M\,(2\,G\,M - c^2\,r)}{c^4\,r^4}$$

$$R_{3,1,1,3} = -\frac{G\,M}{r^2\,(2\,G\,M - c^2\,r)}$$

$$R_{3,1,3,1} = \frac{G\,M}{r^2\,(2\,G\,M - c^2\,r)}$$

$$R_{3,2,2,3} = -\frac{2\,G\,M}{c^2\,r}$$

$$R_{3,2,3,2} = \frac{2\,G\,M}{c^2\,r}$$

(%i13)
```
/* Ricci tensor Ric[mu,nu] */
for mu:0 thru 3 do {
for nu:0 thru 3 do {
    Ric[mu,nu]: sum(R[lambda,mu,lambda,nu], lambda, 0, 3)
}}$
```

(%i14)
```
/* dislplay Ric's being different from zero */
for i:0 thru 3 do {
for j:0 thru 3 do {
    Ric[i,j]  : /*ratsimp*/(factor(Ric[i,j])),
    if Ric[i,j] # 0 then display(Ric[i,j])
    }}$
```

(%i15)
```
/* Ricci Scalar */
RicSc: sum(gContr[0,lambda]*Ric[lambda,0], lambda, 0, 3)
    +  sum(gContr[1,lambda]*Ric[lambda,1], lambda, 0, 3)
    +  sum(gContr[2,lambda]*Ric[lambda,2], lambda, 0, 3)
    +  sum(gContr[3,lambda]*Ric[lambda,3], lambda, 0, 3)
;
```

(%o15) 0

(%i16)
```
ratsimp(RicSc);
```

(%o16) 0

(%i17)
```
/* Raising of indices,
   contravarinat metric el. is g^x^x(contr.) = 1/g_x_x(cov.) */
/*print("Riemann elements R^0_1^0^1, R^0_2^0^2, R^0_3^0^3:");*/

R0101: f(0,1);
R0202: f(0,2);
R0303: f(0,3);
```

(%o17) $\frac{2\,G\,M}{r^2\,(2\,G\,M - c^2\,r)}$

(%o18) $-\frac{G\,M}{r^2\,(2\,G\,M - c^2\,r)}$

(%o19) $$-\frac{G\,M}{r^2\,(2\,G\,M-c^2\,r)}$$

(%i20)
```
R0101: factor(R0101);
R0202: factor(R0202);
R0303: factor(R0303);
```

(%o20) $$\frac{2\,G\,M}{r^2\,(2\,G\,M-c^2\,r)}$$

(%o21) $$-\frac{G\,M}{r^2\,(2\,G\,M-c^2\,r)}$$

(%o22) $$-\frac{G\,M}{r^2\,(2\,G\,M-c^2\,r)}$$

(%i23)
```
R1010: f(1,0);
R1212: f(1,2);
R1313: f(1,3);
```

(%o23) $$-\frac{2\,G\,M\,(2\,G\,M-c^2\,r)}{c^4\,r^4}$$

(%o24) $$\frac{G\,M\,(2\,G\,M-c^2\,r)}{c^4\,r^4}$$

(%o25) $$\frac{G\,M\,(2\,G\,M-c^2\,r)}{c^4\,r^4}$$

(%i26)
```
R1010: factor(R1010);
R1212: factor(R1212);
R1313: factor(R1313);
```

(%o26) $$-\frac{2\,G\,M\,(2\,G\,M-c^2\,r)}{c^4\,r^4}$$

(%o27) $$\frac{G\,M\,(2\,G\,M-c^2\,r)}{c^4\,r^4}$$

(%o28) $$\frac{G\,M\,(2\,G\,M-c^2\,r)}{c^4\,r^4}$$

(%i29)
```
R2020: f(2,0);
R2121: f(2,1);
R2323: f(2,3);
```

(%o29) $$-\frac{G\,M}{c^2\,r^5}$$

(%o30) $$-\frac{G\,M}{c^2\,r^5}$$

(%o31) $\frac{2\,G\,M}{c^2\,r^5}$

(%i32)
```
R2020: factor(R2020);
R2121: factor(R2121);
R2323: factor(R2323);
```

(%o32) $-\frac{G\,M}{c^2\,r^5}$

(%o33) $-\frac{G\,M}{c^2\,r^5}$

(%o34) $\frac{2\,G\,M}{c^2\,r^5}$

(%i35)
```
R3030: f(3,0);
R3131: f(3,1);
R3232: f(3,2);
```

(%o35) $-\frac{G\,M}{c^2\,r^5\,sin(\theta)^2}$

(%o36) $-\frac{G\,M}{c^2\,r^5\,sin(\theta)^2}$

(%o37) $\frac{2\,G\,M}{c^2\,r^5\,sin(\theta)^2}$

(%i38)
```
R3030: factor(R3030);
R3131: factor(R3131);
R3232: factor(R3232);
```

(%o38) $-\frac{G\,M}{c^2\,r^5\,sin(\theta)^2}$

(%o39) $-\frac{G\,M}{c^2\,r^5\,sin(\theta)^2}$

(%o40) $\frac{2\,G\,M}{c^2\,r^5\,sin(\theta)^2}$

(%i41)
```
/* Coulomb law */
DivE : R0101 + R0202 + R0303;
```

(%o41) 0

(%i42)
```
/* J[r] */
Jr : -(R1010 + R1212 + R1313);
```

(%o42) 0

(%i43)
```
ratsimp(Jr);
```

```
(%o43)  0

(%i44)  /* J[theta] */
        Jtheta : -(R2020 + R2121 + R2323);
(%o44)  0

(%i45)  ratsimp(Jtheta);
(%o45)  0

(%i46)  /* J[phi] */
        Jphi : -(R3030 + R3131 + R3232);
(%o46)  0

(%i47)  ratsimp(Jphi);
(%o47)  0

(%i48)
```