(%i1)
```
/* define special summation function */
f(i,j) := sum(R[i,j,sigma,0]*gContr[i,sigma]*gContr[j,0],sigma,0,3)
        + sum(R[i,j,sigma,1]*gContr[i,sigma]*gContr[j,1],sigma,0,3)
        + sum(R[i,j,sigma,2]*gContr[i,sigma]*gContr[j,2],sigma,0,3)
        + sum(R[i,j,sigma,3]*gContr[i,sigma]*gContr[j,3],sigma,0,3);
```
(%o1) $$f(i,j):=\mathrm{sum}(R_{i,j,\sigma,0}\,gContr_{i,\sigma}\,gContr_{j,0},\sigma,0,3)+\mathrm{sum}(R_{i,j,\sigma,1}\,gContr_{i,\sigma}\,gContr_{j,1},\sigma,0,3)+\mathrm{sum}(R_{i,j,\sigma,2}\,gContr_{i,\sigma}\,gContr_{j,2},\sigma,0,3)+\mathrm{sum}(R_{i,j,\sigma,3}\,gContr_{i,\sigma}\,gContr_{j,3},\sigma,0,3)$$

(%i2)
```
/* define coordinate vector */
array(x, 3);
[x[0],x[1],x[2],x[3]]: [t, r, theta, phi];
```
(%o2) x

(%o3) $[\,t\,,\,r\,,\,\theta\,,\,\varphi\,]$

(%i4)
```
/* define coordinate dependent functions */
depends([a], [t]);
```
(%o4) $[\,a(t)\,]$

(%i5)
```
/* g1 is symm. metric with indices 1...4 */
g1: matrix(
 [-1,0,0,0],
 [0,a^2/(1-k*r^2),0,0],
 [0,0,a^2*r^2,0],
 [0,0,0,a^2*r^2*sin(theta)^2]
);
```
(%o5)
$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-k\,r^2} & 0 & 0 \\ 0 & 0 & a^2\,r^2 & 0 \\ 0 & 0 & 0 & a^2\,r^2\,\sin(\theta)^2 \end{bmatrix}$$

(%i6)
```
/* contravariant g is inverse of g */
gContr1: ratsimp(invert(g1));
```

(%o6)
$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -\frac{k\,r^2-1}{a^2} & 0 & 0 \\ 0 & 0 & \frac{1}{a^2\,r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{a^2\,r^2\,\sin(\theta)^2} \end{bmatrix}$$

(%i7)
```
/* g1 and gContr1 are transformed to g and gContr (indices 0...3) */
for mu:0 thru 3 do {
for nu:0 thru 3 do {
    g     [mu,nu]: g1     [mu+1, nu+1],
    gContr[mu,nu]: gContr1[mu+1, nu+1]
}}$
```

(%i8)
```
/* computation of Christoffel symbols Gamma^sigma_mu_nu */
for sigma:0 thru 3 do {
for mu:0 thru 3 do {
for nu:0 thru 3 do {
    Gamma[sigma,mu,nu] :
    /* rho sum by function call: */
    sum(
        1/2 * gContr[sigma,rho]*(
        diff(g[nu,rho],x[mu] ) +
        diff(g[rho,mu],x[nu] ) -
        diff(g[mu,nu] ,x[rho])),
    rho, 0, 3),
    /* evaluate differentiation dy/dr */
    Gamma[sigma,mu,nu]: ev(Gamma[sigma,mu,nu],diff)
}}}$
```

(%i9)
```
/* dislplay Gamma's being different from zero */
for i:0 thru 3 do {
for j:0 thru 3 do {
for k:0 thru 3 do {
    if Gamma[i,j,k] # 0 then {
        display(Gamma[i,j,k])
    }}}}$
```

$$\Gamma_{0,1,1} = \frac{a\left(\frac{d}{d\,t}a\right)}{1-k\,r^2}$$

$$\Gamma_{0,2,2} = a\left(\frac{d}{d\,t}a\right) r^2$$

$$\Gamma_{0,3,3} = a\left(\frac{d}{d\,t}a\right) r^2\,\sin(\theta)^2$$

$$\Gamma_{1,0,1} = -\frac{\left(\frac{d}{dt}a\right)(k r^2 - 1)}{a(1 - k r^2)}$$

$$\Gamma_{1,1,0} = -\frac{\left(\frac{d}{dt}a\right)(k r^2 - 1)}{a(1 - k r^2)}$$

$$\Gamma_{1,1,1} = -\frac{k r (k r^2 - 1)}{(1 - k r^2)^2}$$

$$\Gamma_{1,2,2} = r (k r^2 - 1)$$

$$\Gamma_{1,3,3} = r (k r^2 - 1) \sin(\theta)^2$$

$$\Gamma_{2,0,2} = \frac{\frac{d}{dt}a}{a}$$

$$\Gamma_{2,1,2} = \frac{1}{r}$$

$$\Gamma_{2,2,0} = \frac{\frac{d}{dt}a}{a}$$

$$\Gamma_{2,2,1} = \frac{1}{r}$$

$$\Gamma_{2,3,3} = -\cos(\theta)\sin(\theta)$$

$$\Gamma_{3,0,3} = \frac{\frac{d}{dt}a}{a}$$

$$\Gamma_{3,1,3} = \frac{1}{r}$$

$$\Gamma_{3,2,3} = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\Gamma_{3,3,0} = \frac{\frac{d}{dt}a}{a}$$

$$\Gamma_{3,3,1} = \frac{1}{r}$$

$$\Gamma_{3,3,2} = \frac{\cos(\theta)}{\sin(\theta)}$$

```
(%i10)  /* compute Riemann tensor elements */
        for rho:0 thru 3 do {
        for sigma:0 thru 3 do {
        for mu:0 thru 3 do {
        for nu:0 thru 3 do {
            R[rho,sigma,mu,nu] :
            diff(Gamma[rho,nu,sigma],x[mu]) -
            diff(Gamma[rho,mu,sigma],x[nu]) +
            /* lambda sums by function call: */
            sum(
                Gamma[rho,mu,lambda] * Gamma[lambda,nu,sigma] -
                Gamma[rho,nu,lambda] * Gamma[lambda,mu,sigma],
                lambda, 0, 3)
        }}}}$
```

```
(%i11)  /* dislplay R's being different from zero */
        for i:0 thru 3 do {
        for j:0 thru 3 do {
        for k:0 thru 3 do {
        for l:0 thru 3 do {
            R[i,j,k,l] : /*ratsimp*/(factor(R[i,j,k,l])),
            if R[i,j,k,l] # 0 then display(R[i,j,k,l])
            }}}}$
```

$$R_{0,1,0,1} = -\frac{a\left(\frac{d^2}{dt^2}a\right)}{k r^2 - 1}$$

$$R_{0,1,1,0} = \frac{a\left(\frac{d^2}{dt^2}a\right)}{k r^2 - 1}$$

$$R_{0,2,0,2} = a\left(\frac{d^2}{dt^2}a\right) r^2$$

$$R_{0,2,2,0} = -a\left(\frac{d^2}{dt^2}a\right) r^2$$

$$R_{0,3,0,3} = a\left(\frac{d^2}{dt^2}a\right) r^2 \, sin(\theta)^2$$

$$R_{0,3,3,0} = -a\left(\frac{d^2}{dt^2}a\right) r^2 \, sin(\theta)^2$$

$$R_{1,0,0,1} = \frac{\frac{d^2}{dt^2}a}{a}$$

$$R_{1,0,1,0} = -\frac{\frac{d^2}{dt^2}a}{a}$$

$$R_{1,2,1,2} = \left(k + \left(\frac{\mathrm{d}}{\mathrm{d}\,t}a\right)^2\right) r^2$$

$$R_{1,2,2,1} = -\left(k + \left(\frac{\mathrm{d}}{\mathrm{d}\,t}a\right)^2\right) r^2$$

$$R_{1,3,1,3} = \left(k + \left(\frac{\mathrm{d}}{\mathrm{d}\,t}a\right)^2\right) r^2 \, sin(\theta)^2$$

$$R_{1,3,3,1} = -\left(k + \left(\frac{\mathrm{d}}{\mathrm{d}\,t}a\right)^2\right) r^2 \, sin(\theta)^2$$

$$R_{2,0,0,2} = \frac{\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}a}{a}$$

$$R_{2,0,2,0} = -\frac{\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}a}{a}$$

$$R_{2,1,1,2} = \frac{k + \left(\frac{\mathrm{d}}{\mathrm{d}\,t}a\right)^2}{k\,r^2 - 1}$$

$$R_{2,1,2,1} = -\frac{k + \left(\frac{\mathrm{d}}{\mathrm{d}\,t}a\right)^2}{k\,r^2 - 1}$$

$$R_{2,3,2,3} = \left(k + \left(\frac{\mathrm{d}}{\mathrm{d}\,t}a\right)^2\right) r^2 \, sin(\theta)^2$$

$$R_{2,3,3,2} = -\left(k + \left(\frac{\mathrm{d}}{\mathrm{d}\,t}a\right)^2\right) r^2 \, sin(\theta)^2$$

$$R_{3,0,0,3} = \frac{\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}a}{a}$$

$$R_{3,0,3,0} = -\frac{\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}a}{a}$$

$$R_{3,1,1,3} = \frac{k + \left(\frac{\mathrm{d}}{\mathrm{d}\,t}a\right)^2}{k\,r^2 - 1}$$

$$R_{3,1,3,1} = -\frac{k + \left(\frac{\mathrm{d}}{\mathrm{d}\,t}a\right)^2}{k\,r^2 - 1}$$

$$R_{3,2,2,3} = -\left(k + \left(\frac{\mathrm{d}}{\mathrm{d}\,t}a\right)^2\right) r^2$$

$$R_{3,2,3,2} = \left(k + \left(\frac{d}{dt}a\right)^2\right) r^2$$

(%i12)
```
/* Ricci tensor Ric[mu,nu] */
for mu:0 thru 3 do {
for nu:0 thru 3 do {
    Ric[mu,nu]: sum(R[lambda,mu,lambda,nu], lambda, 0, 3)
}}$
```

(%i13)
```
/* dislplay Ric's being different from zero */
for i:0 thru 3 do {
for j:0 thru 3 do {
    Ric[i,j] : /*ratsimp*/(factor(Ric[i,j])),
    if Ric[i,j] # 0 then display(Ric[i,j])
    }}$
```

$$Ric_{0,0} = -\frac{3\left(\frac{d^2}{dt^2}a\right)}{a}$$

$$Ric_{1,1} = -\frac{2k + a\left(\frac{d^2}{dt^2}a\right) + 2\left(\frac{d}{dt}a\right)^2}{kr^2 - 1}$$

$$Ric_{2,2} = \left(2k + a\left(\frac{d^2}{dt^2}a\right) + 2\left(\frac{d}{dt}a\right)^2\right) r^2$$

$$Ric_{3,3} = \left(2k + a\left(\frac{d^2}{dt^2}a\right) + 2\left(\frac{d}{dt}a\right)^2\right) r^2 \, sin(\theta)^2$$

(%i14)
```
/* Ricci Scalar */
RicSc: sum(gContr[0,lambda]*Ric[lambda,0], lambda, 0, 3)
    +  sum(gContr[1,lambda]*Ric[lambda,1], lambda, 0, 3)
    +  sum(gContr[2,lambda]*Ric[lambda,2], lambda, 0, 3)
    +  sum(gContr[3,lambda]*Ric[lambda,3], lambda, 0, 3)
;
```

(%o14)
$$\frac{3\left(2k + a\left(\frac{d^2}{dt^2}a\right) + 2\left(\frac{d}{dt}a\right)^2\right)}{a^2} + \frac{3\left(\frac{d^2}{dt^2}a\right)}{a}$$

(%i15)
```
ratsimp(RicSc);
```

(%o15)
$$\frac{6k + 6a\left(\frac{d^2}{dt^2}a\right) + 6\left(\frac{d}{dt}a\right)^2}{a^2}$$

(%i16)

```
/* Test for R^q */
for mu:   0 thru 3 do (
for sigma:0 thru 3 do (
for nu:   0 thru 3 do (
for rho:  0 thru 3 do (
    R_q: R[mu,sigma,nu,rho] + R[mu,rho,sigma,nu] + R[mu,nu,rho,sigma],
    if R_q # 0 then (
        display("=============Einstein equation R^q=0 not fulfilled! "),
        display(mu,sigma,nu,rho),
        display(R_q)
    )
))));
```

(%o16) *done*

```
(%i17) /* Raising of indices,
          contravarinat metric el. is g^x^x(contr.) = 1/g_x_x(cov.) */
       /*print("Riemann elements R^0_1^0^1, R^0_2^0^2, R^0_3^0^3:");*/

       R0101: f(0,1);
       R0202: f(0,2);
       R0303: f(0,3);
```

$$(\%o17)\quad -\frac{\frac{d^2}{dt^2}a}{a}$$

$$(\%o18)\quad -\frac{\frac{d^2}{dt^2}a}{a}$$

$$(\%o19)\quad -\frac{\frac{d^2}{dt^2}a}{a}$$

```
(%i20) R0101: factor(R0101);
       R0202: factor(R0202);
       R0303: factor(R0303);
```

$$(\%o20)\quad -\frac{\frac{d^2}{dt^2}a}{a}$$

$$(\%o21)\quad -\frac{\frac{d^2}{dt^2}a}{a}$$

$$(\%o22)\quad -\frac{\frac{d^2}{dt^2}a}{a}$$

```
(%i23) R1010: f(1,0);
       R1212: f(1,2);
       R1313: f(1,3);
```

$$(\%o23)\quad -\frac{\left(\frac{d^2}{dt^2}a\right)(k\,r^2-1)}{a^3}$$

$$(\%o24)\quad -\frac{\left(k+\left(\frac{d}{dt}a\right)^2\right)(k\,r^2-1)}{a^4}$$

$$(\%o25)\quad -\frac{\left(k+\left(\frac{d}{dt}a\right)^2\right)(k\,r^2-1)}{a^4}$$

```
(%i26) R1010: factor(R1010);
       R1212: factor(R1212);
       R1313: factor(R1313);
```

$$(\%o26)\quad -\frac{\left(\frac{d^2}{dt^2}a\right)(k\,r^2-1)}{a^3}$$

$$(\%o27)\quad -\frac{\left(k+\left(\frac{d}{dt}a\right)^2\right)(k\,r^2-1)}{a^4}$$

$$(\%o28)\quad -\frac{\left(k+\left(\frac{d}{dt}a\right)^2\right)(k\,r^2-1)}{a^4}$$

```
(%i29) R2020: f(2,0);
       R2121: f(2,1);
       R2323: f(2,3);
```

$$(\%o29)\quad \frac{\frac{d^2}{dt^2}a}{a^3\,r^2}$$

$$(\%o30)\quad \frac{k+\left(\frac{d}{dt}a\right)^2}{a^4\,r^2}$$

$$(\%o31)\quad \frac{k+\left(\frac{d}{dt}a\right)^2}{a^4\,r^2}$$

```
(%i32) R2020: factor(R2020);
       R2121: factor(R2121);
       R2323: factor(R2323);
```

$$(\%o32)\quad \frac{\frac{d^2}{dt^2}a}{a^3\,r^2}$$

(%o33)
$$\frac{k+\left(\frac{\mathrm{d}}{\mathrm{d}\,t}a\right)^2}{a^4\,r^2}$$

(%o34)
$$\frac{k+\left(\frac{\mathrm{d}}{\mathrm{d}\,t}a\right)^2}{a^4\,r^2}$$

(%i35)
```
R3030: f(3,0);
R3131: f(3,1);
R3232: f(3,2);
```

(%o35)
$$\frac{\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}a}{a^3\,r^2\,\sin(\theta)^2}$$

(%o36)
$$\frac{k+\left(\frac{\mathrm{d}}{\mathrm{d}\,t}a\right)^2}{a^4\,r^2\,\sin(\theta)^2}$$

(%o37)
$$\frac{k+\left(\frac{\mathrm{d}}{\mathrm{d}\,t}a\right)^2}{a^4\,r^2\,\sin(\theta)^2}$$

(%i38)
```
R3030: factor(R3030);
R3131: factor(R3131);
R3232: factor(R3232);
```

(%o38)
$$\frac{\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}a}{a^3\,r^2\,\sin(\theta)^2}$$

(%o39)
$$\frac{k+\left(\frac{\mathrm{d}}{\mathrm{d}\,t}a\right)^2}{a^4\,r^2\,\sin(\theta)^2}$$

(%o40)
$$\frac{k+\left(\frac{\mathrm{d}}{\mathrm{d}\,t}a\right)^2}{a^4\,r^2\,\sin(\theta)^2}$$

(%i41)
```
/* Coulomb law */
DivE : R0101 + R0202 + R0303;
```

(%o41)
$$-\frac{3\left(\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}a\right)}{a}$$

(%i42)
```
ratsimp(DivE);
```

(%o42) $$-\frac{3\left(\frac{d^2}{dt^2}a\right)}{a}$$

(%i43)
```
/* J[r] */
Jr : -(R1010 + R1212 + R1313);
```

(%o43) $$\frac{2\left(k+\left(\frac{d}{dt}a\right)^2\right)(k\,r^2-1)}{a^4}+\frac{\left(\frac{d^2}{dt^2}a\right)(k\,r^2-1)}{a^3}$$

(%i44)
```
ratsimp(Jr);
```

(%o44) $$\frac{\left(2\,k^2+\left(a\left(\frac{d^2}{dt^2}a\right)+2\left(\frac{d}{dt}a\right)^2\right)k\right)r^2-2\,k-a\left(\frac{d^2}{dt^2}a\right)-2\left(\frac{d}{dt}a\right)^2}{a^4}$$

(%i45)
```
/* J[theta] */
Jtheta : -(R2020 + R2121 + R2323);
```

(%o45) $$-\frac{2\left(k+\left(\frac{d}{dt}a\right)^2\right)}{a^4\,r^2}-\frac{\frac{d^2}{dt^2}a}{a^3\,r^2}$$

(%i46)
```
ratsimp(Jtheta);
```

(%o46) $$-\frac{2\,k+a\left(\frac{d^2}{dt^2}a\right)+2\left(\frac{d}{dt}a\right)^2}{a^4\,r^2}$$

(%i47)
```
/* J[phi] */
Jphi : -(R3030 + R3131 + R3232);
```

(%o47) $$-\frac{2\left(k+\left(\frac{d}{dt}a\right)^2\right)}{a^4\,r^2\,\sin(\theta)^2}-\frac{\frac{d^2}{dt^2}a}{a^3\,r^2\,\sin(\theta)^2}$$

(%i48)
```
ratsimp(Jphi);
```

(%o48) $$-\frac{2\,k+a\left(\frac{d^2}{dt^2}a\right)+2\left(\frac{d}{dt}a\right)^2}{a^4\,r^2\,\sin(\theta)^2}$$

(%i49)
```
a: 1;
```

(%o49) 1

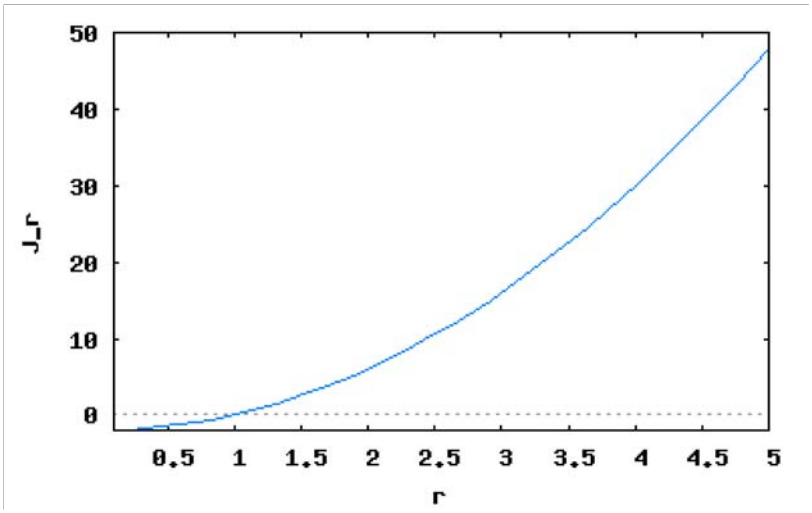
(%i50)
```
DivE_p: ev(at(DivE,[a=1, k=1]));
```
(%o50) 0

(%i51)
```
Jr_p: ev(at(Jr,[a=1, k=1]));
```
(%o51) $2\,(r^2-1)$

(%i52)
```
Jtheta_p: ev(at(Jtheta,[a=1, k=1, theta=%pi/2]));
```
(%o52) $-\frac{2}{r^2}$

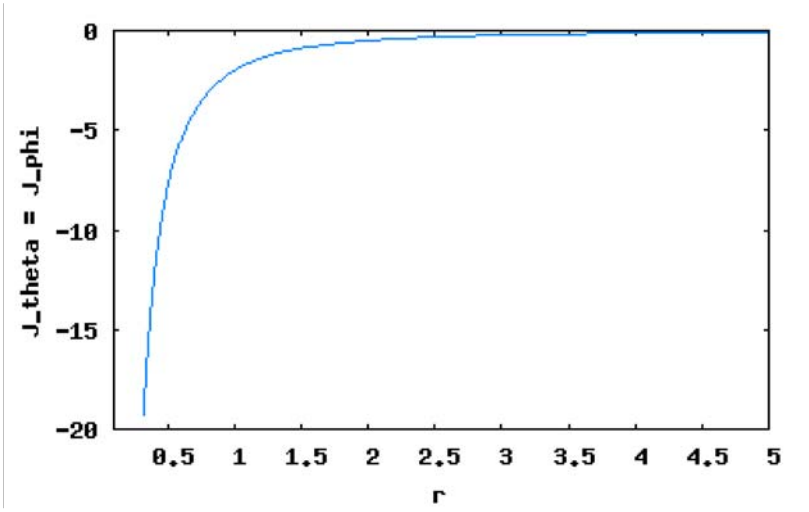
(%i53)
```
wxplot2d([Jr_p], [r,.1,5],[y,-2,50], [gnuplot_preamble, "set zeroaxis;"],
[xlabel, "r"], [ylabel, "J_r"])$
```
Output file "C:/Documents and Settings/Administrator/maxout.png".

(%t53)



(%i54)
```
wxplot2d([Jtheta_p], [r,.1,5],[y,-20,0], [gnuplot_preamble, "set zeroaxis;"],
[xlabel, "r"], [ylabel, "J_theta = J_phi"])$
```
Output file "C:/Documents and Settings/Administrator/maxout.png".

(%t54)



```
(%i55)
```