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BELTRAMI EQUATION FOR $\underline{B}^{(3)}$

"Foundational Electrodynamics and Beltrami Vector Fields." by Donald Reed, 5260 Lake Road, Orchard Park, NY 14127, in T. W. Barrett and D. M. Grimes, "Advanced Electromagnetism" (World Scientific, 1995).

Reed mentions on p. 233 that Beltrami flow could play a major role in foundational electrodynamics, structuring the vacuum fields of nature. In these notes it is shown that the $\underline{B}^{(1)}$, $\underline{B}^{(2)}$ and $\underline{B}^{(3)}$ fields are particular solutions of the solenoidal Beltrami equation (p. 228):

$$\underline{\nabla} \times \underline{B} = k\underline{B}. \quad —(1)$$

Main Results (SI Units), Transverse Plane Waves

1) If:

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}; \quad \underline{\nabla} \times \underline{B} = \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} \quad —(2)$$

then:

$$\left.\begin{aligned} \underline{\nabla} \times \underline{B} &= k\underline{B} \\ \underline{\nabla} \times \underline{E} &= k\underline{E} \\ \underline{\nabla} \times \underline{A} &= k\underline{A} \end{aligned}\right\} \quad —(3)$$

where:

$$\underline{B} = \underline{\nabla} \times \underline{A}, \quad —(4)$$

and where:

$$k = \pm \kappa. \quad —(5)$$

Here k is a pseudoscalar which changes sign between left and right circularly polarized vacuum radiation.

2) The $\underline{B}^{(3)}$ Equation

$$\underline{\nabla} \times \underline{B}^{(3)} = k\underline{B}^{(3)} \quad —(6)$$

$$k = 0$$

where:

$$\underline{B}^{(1)} \times \underline{B}^{(2)} = iB^{(0)}\underline{B}^{(3)*}. \quad —(7)$$

1a):

The Transverse Plane Waves

Left handed

$$\underline{E}_L^{(1)} = \frac{E^{(0)}}{\sqrt{2}}(\underline{i} - i\underline{j})e^{-i(\omega t - \kappa z)} = \underline{E}_L^{(2)*}$$

$$\underline{B}_L^{(1)} = \frac{B^{(0)}}{\sqrt{2}}(i\underline{i} + \underline{j})e^{-i(\omega t - \kappa z)} = \underline{B}_L^{(2)*}$$

$$\underline{A}_L^{(1)} = \frac{A^{(0)}}{\sqrt{2}}(i\underline{i} + \underline{j})e^{-i(\omega t - \kappa z)} = \underline{A}_L^{(2)*}$$

Right handed

$$\underline{E}_R^{(1)} = \frac{E^{(0)}}{\sqrt{2}}(\underline{i} + i\underline{j})e^{-i(\omega t - \kappa z)} = \underline{E}_R^{(2)*}$$

$$\underline{B}_R^{(1)} = \frac{B^{(0)}}{\sqrt{2}}(-i\underline{i} + \underline{j})e^{-i(\omega t - \kappa z)} = \underline{B}_R^{(2)*}$$

$$\underline{A}_R^{(1)} = \frac{A^{(0)}}{\sqrt{2}}(-i\underline{i} + \underline{j})e^{-i(\omega t - \kappa z)} = \underline{A}_R^{(2)*}$$

The Longitudinal $\underline{B}^{(3)}$ Field

$$\underline{B}_L^{(3)} = -\underline{B}_R^{(3)} = B^{(0)}\underline{k}$$

Therefore:

$$\underline{\nabla} \times \underline{B}_L^{(1)} = -\kappa \underline{B}_L^{(1)}; \quad \underline{\nabla} \times \underline{B}_R^{(1)} = +\kappa \underline{B}_R^{(1)}$$

$$\underline{\nabla} \times \underline{E}_L^{(1)} = -\kappa \underline{E}_L^{(1)}; \quad \underline{\nabla} \times \underline{E}_R^{(1)} = +\kappa \underline{E}_R^{(1)}$$

$$\underline{\nabla} \times \underline{A}_L^{(1)} = -\kappa \underline{A}_L^{(1)}; \quad \underline{\nabla} \times \underline{A}_R^{(1)} = +\kappa \underline{A}_R^{(1)}$$

and similarly for index (2).

The Longitudinal Fields

$$\underline{\nabla} \times \underline{B}_R^{(3)} = \underline{\nabla} \times \underline{B}_L^{(3)} = \underline{0}$$

ALL COMPONENTS ARE DESCRIBED BY BELTRAMI EQUATIONS.

11b)

I. Since $\underline{E}$ and $\underline{B}$ are foundational fields these equations are valid under all conditions. They are all force-free equations.

II. The Fundamental Symmetry

a) Eqns. (3) are more general than eqns. (2) because the former are valid for $\underline{B}^{(1)}$, $\underline{B}^{(2)}$ and $\underline{B}^{(3)}$. The latter only for $\underline{B}^{(1)}$ and $\underline{B}^{(2)}$, because the $\underline{B}^{(3)}$ Maxwell equation has a different structure from the same structure equations (2). The $\underline{B}^{(3)}$ equation has the same structure as the Beltrami equations for $\underline{B}^{(1)}$ and $\underline{B}^{(2)}$.

b) Because of the pseudoscalar k in eqns. (3), there are two distinct Beltrami equations for every Maxwell equation, e.g.:

$$\boxed{\underline{\nabla} \times \underline{E}_L^{(1)} = -\frac{\partial \underline{B}_L^{(1)}}{\partial t} \quad \text{and} \quad \underline{\nabla} \times \underline{E}_R^{(1)} = -\frac{\partial \underline{B}_R^{(1)}}{\partial t}}$$

$$\updownarrow$$

$$\boxed{\underline{\nabla} \times \underline{E}_L^{(1)} = -\kappa \underline{E}_L^{(1)} \quad \text{and} \quad \underline{\nabla} \times \underline{E}_R^{(1)} = +\kappa \underline{E}_R^{(1)}}$$

THE FARADAY LAW DOES NOT DISTINGUISH BETWEEN LEFT AND RIGHT CIRCULAR POLARIZATION. THE CORRESPONDING BELTRAMI EQUATIONS ARE DISTINCT EQUATIONS FOR THE TWO DISTINCT STATES OF RADIATION.

More accurately, eqns. (3) are Beltrami / Helmholtz equations.

We can conclude that $\underline{B}^{(3)}$ is a solution of the Beltrami equation with $k = 0$, i.e. $\underline{B}^{(3)}$ is <u>solenoidal</u> <u>and irrotational.</u> It is illustrated in Fig. 8 of Reed, p. 231, as the line on the axis going up the page, <u>and is part of the general solution of the solenoidal Beltrami equation given by S. Chandrasekhar and P. C. Kendall</u>, "On Force Free Magnetic Fields", Astrophys. J., <u>126</u>, 457–460 (1957). There is equipartition between the toroidal and poloidal components as shown by Chandrasekhar.

THIS PROVES THAT $\underline{B}^{(3)}$ $\boxed{\text{IS NON-ZERO}}$ IF WE ACCEPT THE BELTRAMI EQUATION, OR HELMHOLTZ EQUATION, FOR VACUUM RADIATION. THUS, E/M IN VACUO HAS SU(2) OR O(3) SYMMETRY. IN EQN. (7), IF $\underline{B}^{(3)} = ?\ \underline{0}$, THEN $\underline{B}^{(1)}$ AND $\underline{B}^{(2)}$ VANISH.

The general solution given by Chandrasekhar and Kendall is:

$$B = B_0 \left(0, J_1(kr), J_0(kr) \right)$$

where J_1 and J_0 are Bessel functions. As $k \to 0$: $B \to B_0 \left(0, J_1(0), J_0(0) \right)$

$$= (0, 0, B_0).$$

i.e. $\boxed{\underline{B}^{(3)} = B^{(0)} \underline{k} \ \Big|\ \neq \underline{0}}$

(3) GENERAL SOLUTION OF THE BELTRAMI EQUATION

$$\underline{\nabla} \times \underline{B} = k \underline{B} \quad — (8)$$

Refs.

1) A. Konigl and A. R. Choudhuri, "Force Free Equilibria of Magnetic Jets", Astrophys. J., 289, 173 (1985).

2) S. Chandrasekhar and P. C. Kendall, "On Force Free Magnetic Fields", Astrophys. J., 126, 457 (1957).

3) G. Stephenson, "Mathematical Methods for Science Students" (Longmans, London, 1968, fifth printing).

From ref (1), the general solution in cylindrical coordinates of eqn. (8) is:

$$\underline{B} = \sum_{m,n} B_{mn} \, \underline{b}^{mn}(r, \theta, z) \quad — (9)$$

where m is a non-negative integer and where $\underline{b}^{mn}$ depend on θ and z ~~through~~ $\phi = m\theta + nZ$. The expressions for the modes depend on linear combinations of ~~B~~ Bessel and Neumann functions, J_m and N_m, similar to the Helmholtz equation, Jackson, chapt. (16). When the domain of solution involves the axis $r = 0$, and we restrict solutions to an axisymmetric wave equation, then:

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r} \right) + k^2 \psi = 0. \quad — (10)$$

The solution is:

$$\psi = C J_0(kr) \quad — (11)$$

where C is any constant.

Substituting in:

$$\underline{B} = k \underline{\nabla} \times (\psi \underline{a}) + \underline{\nabla} \times (\underline{\nabla} \times (\psi \underline{a})) \quad - (12)$$

$$:= \underline{P} + \underline{Q}$$

(poloidal and toroidal), then:

$$\underline{B} = B_0 (0, J_1(kr), J_0(kr)) \quad - (13)$$

for the mode $m = n = 0$; $\underline{a} = (0, 0, 1)$. Therefore the unit vector $\underline{a} = (0, 0, 1)$ designates the Z axis. Eqn (13) was originally derived in ref. (2).

THE $\underline{B}^{(3)}$ EQUATION

The $B^{(3)}$ equation is a Beltrami equation with $k = 0$:

$$\boxed{\underline{\nabla} \times \underline{B}^{(3)} = k \underline{B}^{(3)} \quad (k = 0)} \quad - (14)$$

The solution of eqn (14) is:

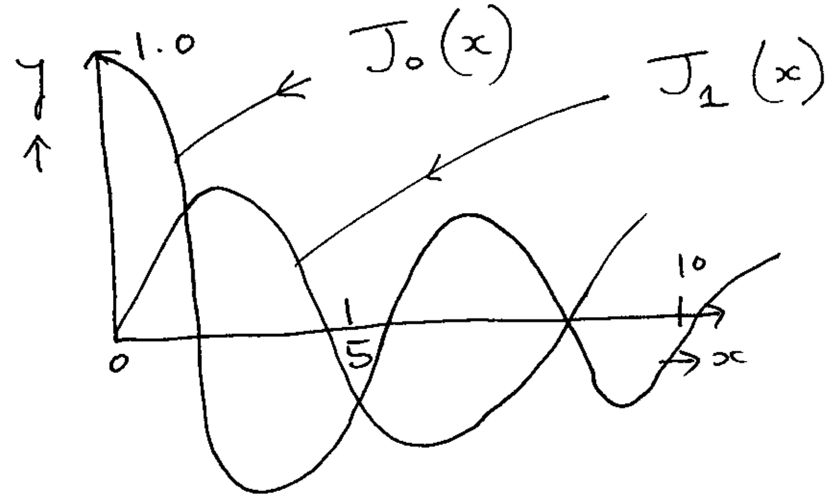
$$\underline{B}^{(3)} = B_0 (0, J_1(0), J_0(0)) \quad - (15)$$

and depends on the Bessel functions $J_1(0)$ and $J_0(0)$.

The Bessel Functions (Ref. (3))

$$J_\nu(x) := \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \, \Gamma(\nu + r + 1)} \left(\frac{x}{2}\right)^{\nu + 2r} \quad - (16)$$

where $\Gamma(\nu + r + 1)$ is a gamma function. The Bessel functions are as follows:



The Bessel functions, $n = 0$, $n = 1$

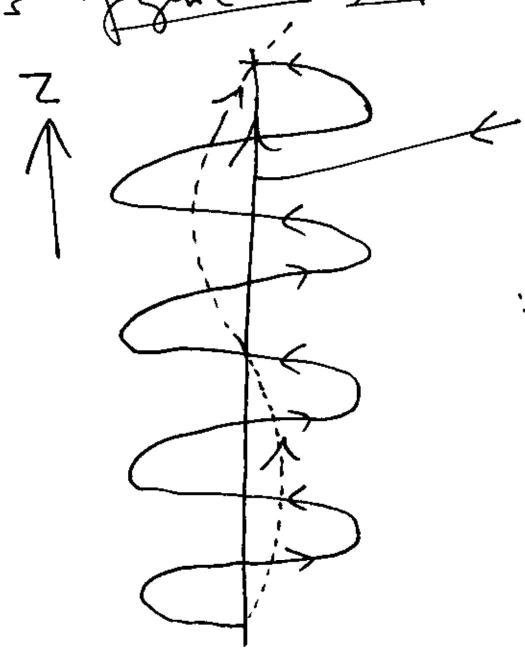
Therefore: $$J_0(0) = 1 \qquad J_0(1) = 0 \tag{17}$$

and: $$\boxed{\begin{aligned} \underline{B}^{(3)} &= \underline{B}(k = 0, m = 0, n = 0) \\ &= B_0(0, 0, 1) \\ &= B_0 \underset{\sim}{a} := B^{(0)} \underline{k} \end{aligned}} \tag{18}$$

Sketch of Solutions (Solenoidal, Axisymmetric)

This is given in Reed's figure 8, p. 231:

> The $\underline{B}^{(3)}$ solution is along $\underline{a} \equiv \underline{k}$ and is $\boxed{\text{non-zero}}$ if $B_0 \equiv B^{(0)}$ is non-zero.



$\underline{B}^{(3)} = B_0 \underline{a} := B^{(0)} \underline{k}$

6)

CONCLUSIONS

1) The $\underline{B}^{(3)}$ equation of SU(2) electrodynamics is a Beltrami equation with $\underline{k} = 0$. The solution:

$$\boxed{\underline{B}^{(3)} = B^{(0)}\underline{k}}$$

is a solenoidal, axisymmetric, toroidal solution. Unless $B^{(0)}$ is zero (no radiation) it is rigorously non-zero. The $\underline{B}^{(3)}$ field is phaseless and irrotational and is the fundamental spin of the electromagnetic field.

2) If $\underline{B}^{(1)} = \underline{B}^{(2)*} = \text{Re}\,\underline{B}^{(1)} + i\,\text{Im}\,\underline{B}^{(1)}$

~~the~~ $\underline{B}^{(1)} = \underline{B}^{(2)*}$ is a solution of the Beltrami equation with ~~imaginary~~ ~~complex~~ real $\underline{k}$. The real components $\text{Re}(\underline{B}^{(1)})$ and $\text{Im}(\underline{B}^{(1)})$ are solutions of the Beltrami equation with real $\underline{k}$. The Beltrami equation is an SU(2) field

3) The Beltrami equation is an angular momentum eigenequation: an equation with:

$$\boxed{\underline{B}^{(1)} \times \underline{B}^{(2)} = i B^{(0)} \underline{B}^{(3)*}}.$$

4) As discussed by Reed, the Beltrami equation is an equation of SU(2) electrodynamics. The fields $\underline{B}^{(1)}$, $\underline{B}^{(2)}$ and $\underline{B}^{(3)}$ are eigenfunctions of the Beltrami equation in vacuo.

SUMMARY OF BELTRAMI EQUATIONS

$$\boxed{\begin{aligned} \underline{\nabla} \times \underline{B}^{(1)} &= -\kappa \underline{B}^{(1)} \quad &— (1)\\ \underline{\nabla} \times \underline{B}^{(2)} &= -\kappa \underline{B}^{(2)} \quad &— (2)\\ \underline{\nabla} \times \underline{B}^{(3)} &= 0\,\kappa \underline{B}^{(3)} \quad &— (3) \end{aligned}}$$

Eigenvalues

$$\boxed{\begin{array}{c} -1\,\kappa \\ -1\,\kappa \\ 0\,\kappa \end{array}}$$

Let:
$$\underline{B}^{(1)} = \mathrm{Re}\,\underline{B}^{(1)} + i\,\mathrm{Im}\,\underline{B}^{(1)}$$

then:
$$\boxed{\begin{aligned} \underline{\nabla} \times \left(\mathrm{Re}\,\underline{B}^{(1)}\right) &= -\kappa\left(\mathrm{Re}\,\underline{B}^{(1)}\right) \quad &— (4)\\ \underline{\nabla} \times \left(\mathrm{Re}\,\underline{B}^{(2)}\right) &= -\kappa\left(\mathrm{Re}\,\underline{B}^{(2)}\right) \quad &— (5)\\ \underline{\nabla} \times \left(\mathrm{Re}\,\underline{B}^{(3)}\right) &= 0\left(\mathrm{Re}\,\underline{B}^{(3)}\right) \quad &— (6) \end{aligned}}$$

because:
$$\mathrm{Re}\,\underline{B}^{(3)} = \underline{B}^{(3)}$$

We have:
$$\underline{B}^{(1)} = \frac{B^{(0)}}{\sqrt{2}}\left(i\underline{i} + \underline{j}\right)e^{-i\phi} \quad \text{(plane wave)}$$
$$\mathrm{Re}\,\underline{B}^{(1)} = \frac{B^{(0)}}{\sqrt{2}}\left(\underline{i}\sin\phi + \underline{j}\cos\phi\right) = \mathrm{Re}\,\underline{B}^{(2)}$$
$$\mathrm{Im}\,\underline{B}^{(1)} = \frac{B^{(0)}}{\sqrt{2}}\left(\underline{i}\cos\phi - \underline{j}\sin\phi\right) = -\mathrm{Im}\,\underline{B}^{(2)}$$

and so:
$$\boxed{\begin{aligned} \underline{\nabla} \times \left(\mathrm{Im}\,\underline{B}^{(1)}\right) &= -\kappa\,\mathrm{Im}\,\underline{B}^{(1)} \quad &— (7)\\ \underline{\nabla} \times \left(\mathrm{Im}\,\underline{B}^{(2)}\right) &= -\kappa\,\mathrm{Im}\,\underline{B}^{(2)} \quad &— (8)\\ \mathrm{Im}\,\underline{B}^{(3)} &= 0 \quad &— (9)\\ \text{formally: } \underline{\nabla} \times \left(\mathrm{Im}\,\underline{B}^{(3)}\right) &= 0\left(\mathrm{Im}\,\underline{B}^{(3)}\right) \end{aligned}}$$

(8) COMMENTS

(1) The vacuum fields of e/m are governed by Beltrami equations. The magnetic components form an eigenequation, eqns (1) to (3), which is an angular momentum eigenequation.

(2) The fields $\underline{B}^{(1)}$ and $\underline{B}^{(2)}$ are solutions of the Maxwell equations in vacuo:

SI units

$$\boxed{\begin{aligned} \underline{\nabla} \times \underline{B}^{(1)} &= \frac{1}{c^2} \frac{\partial \underline{E}^{(1)}}{\partial t} \qquad (10) \\ \underline{\nabla} \times \underline{B}^{(2)} &= \frac{1}{c^2} \frac{\partial \underline{E}^{(2)}}{\partial t} \qquad (11) \end{aligned}}$$

Using:

$$\left.\begin{aligned} \underline{E}^{(1)} = \underline{E}^{(2)*} &= \frac{E^{(0)}}{\sqrt{2}} (\underline{i} - i \underline{j}) e^{-i\phi} \\ \phi := \omega t - \kappa Z \quad &; \quad \kappa = \omega / c \end{aligned}\right\} \quad (12)$$

then:

$$\frac{1}{c^2} \frac{\partial \underline{E}^{(1)}}{\partial t} = -\kappa \underline{B}^{(1)} \qquad (13)$$

$$\frac{1}{c^2} \frac{\partial \underline{E}^{(2)}}{\partial t} = -\kappa \underline{B}^{(2)} \qquad (14)$$

so

$$\boxed{\begin{aligned} &\text{Beltrami eqns. (1) and (2)} \\ \equiv \; &\text{Maxwell eqns. (10) and (11)} \end{aligned}}$$

(3) The third Beltrami equation (3) has no equivalent

in Maxwellian theory, so we see the transition from $U(1)$ e/m to $SU(2)$ e/m very clearly.

THE FUNDAMENTAL EQUATIONS OF $SU(2)$ E/M.

NB

$$\boxed{\begin{array}{ll} \underline{\nabla} \times \underline{B} = \kappa \underline{B} & \text{— (1)} \\ \underline{B}^{(1)} \times \underline{B}^{(2)} = i B^{(0)} \underline{B}^{(3)*} & \text{— (2)} \end{array}}$$

The equations of motion of $\underline{B}^{(1)}$, $\underline{B}^{(2)}$ and $\underline{B}^{(3)}$ are Beltrami equations in vacuo.

THE INHERENT PARADOX OF $U(1)$ E/M

In $U(1)$ electrodynamics, $\boxed{\underline{B}^{(3)} = ?\ \underline{0}}$ by arbitrary definition. However, the general solution of eqn (1) leads to $\boxed{\underline{B}^{(3)} = B^{(0)} \underline{\kappa} \neq \underline{0}}$ as we have seen. This is a serious flaw because $\underline{B}^{(1)} = \underline{B}^{(2)*}$ is also a solution of eqn (1), which for $\underline{B}^{(1)} = \underline{B}^{(2)*}$ is the Maxwell equation in vacuo for $\underline{B}^{(1)} = \underline{B}^{(2)*}$.

So, in $U(1)$ electrodynamics, the observable $\underline{B}^{(1)} \times \underline{B}^{(2)}$ is zero, contradicting empirical data (the inverse Faraday effect and RFR).