SU(2) ELECTRODYNAMICS: HOMOGENEOUS FIELD EQNS.

$$\boxed{D_\mu \underline{\tilde{G}}^{\mu\nu} := \underline{0}} \quad \text{— (1)}$$

This is the Feynman / Jacobi identity that gives the SU(2) Gauss and Faraday Laws. We work in the O(3) basis:

$$\left.\begin{aligned} \underline{e}^{(1)} \times \underline{e}^{(2)} &= i\,\underline{e}^{(3)*} \\ \underline{e}^{(2)} \times \underline{e}^{(3)} &= i\,\underline{e}^{(1)*} \\ \underline{e}^{(3)} \times \underline{e}^{(1)} &= i\,\underline{e}^{(2)*} \end{aligned}\right] \quad \text{— (2)}$$

Following Ryder, eqn. (3.155), eqn. (1) is:

$$\partial_\mu \underline{\tilde{G}}^{\mu\nu} + g\,\underline{A}_\mu \times \underline{\tilde{G}}^{\mu\nu} := \underline{0} \quad \text{— (3)}$$

In basis (2):

$$\left.\begin{aligned} \partial_\mu \underline{\tilde{G}}^{\mu\nu(1)*} &:= ig\,\underline{A}_\mu^{(2)} \times \underline{\tilde{G}}^{\mu\nu(3)} \\ \partial_\mu \underline{\tilde{G}}^{\mu\nu(2)*} &:= ig\,\underline{A}_\mu^{(3)} \times \underline{\tilde{G}}^{\mu\nu(1)} \\ \partial_\mu \underline{\tilde{G}}^{\mu\nu(3)*} &:= ig\,\underline{A}_\mu^{(1)} \times \underline{\tilde{G}}^{\mu\nu(2)} \end{aligned}\right] \quad \text{— (4)}$$

In vector notation the Gauss and Faraday Laws emerge from eqn. (4) as follows:

Gauss Law in O(3) Symmetry

$$\boxed{\begin{aligned} \underline{\nabla} \cdot \underline{B}^{(1)*} &:= ig\left(\underline{B}^{(3)} \cdot \underline{A}^{(2)} - \underline{A}^{(3)} \cdot \underline{B}^{(2)}\right) \\ \underline{\nabla} \cdot \underline{B}^{(2)*} &:= ig\left(\underline{B}^{(1)} \cdot \underline{A}^{(3)} - \underline{A}^{(1)} \cdot \underline{B}^{(3)}\right) \\ \underline{\nabla} \cdot \underline{B}^{(3)*} &:= ig\left(\underline{B}^{(2)} \cdot \underline{A}^{(1)} - \underline{A}^{(2)} \cdot \underline{B}^{(1)}\right) \end{aligned}}$$

(5)

In general the $O(3)$ symmetry Gauss Law allows a magnetic monopole. This is zero only in the special case:

$$\underline{B}^{(3)} \cdot \underline{A}^{(2)} = \underline{A}^{(3)} \cdot \underline{B}^{(2)} \quad - (6a)$$

$$\underline{B}^{(1)} \cdot \underline{A}^{(3)} = \underline{A}^{(1)} \cdot \underline{B}^{(3)} \quad - (6b)$$

$$\underline{B}^{(2)} \cdot \underline{A}^{(1)} = \underline{A}^{(2)} \cdot \underline{B}^{(1)} \quad - (6c)$$

If $\underline{B}^{(1)} = \underline{B}^{(2)*} = \underline{\nabla} \times \underline{A}^{(1)} = \underline{\nabla} \times \underline{A}^{(2)*}$ is a transverse plane wave, then eqn. (6c) is satisfied automatically. Eqns. (6a) and (6b) imply:

$$\boxed{|\underline{B}^{(3)}| = \kappa \, |\underline{A}^{(3)}|} \quad - (7)$$

and if $\underline{B}^{(3)} \neq \underline{0}$; $\underline{A}^{(3)} \neq \underline{0}$. Since $\underline{B}^{(3)}$ is non-zero by definition in $O(3)$ electrodynamics, so is $\underline{A}^{(3)}$. The complete $A^{\mu(3)}$ four-vector is therefore fully covariant:

$$A^{\mu(3)} = \left(A_0^{(3)}, \underline{A}^{(3)} \right) \quad - (8)$$

Faraday Law in $O(3)$ Symmetry

$$\boxed{\begin{aligned}
\underline{\nabla} \times \underline{E}^{(1)*} + \frac{\partial \underline{B}^{(1)*}}{\partial t} &= -ig \left(c A_0^{(3)} \underline{B}^{(2)} - c A_0^{(2)} \underline{B}^{(3)} + \underline{A}^{(2)} \times \underline{E}^{(3)} - \underline{A}^{(3)} \times \underline{E}^{(2)} \right) \\
\underline{\nabla} \times \underline{E}^{(2)*} + \frac{\partial \underline{B}^{(2)*}}{\partial t} &= -ig \left(c A_0^{(1)} \underline{B}^{(3)} - c A_0^{(3)} \underline{B}^{(1)} + \underline{A}^{(3)} \times \underline{E}^{(1)} - \underline{A}^{(1)} \times \underline{E}^{(3)} \right) \\
\underline{\nabla} \times \underline{E}^{(3)*} + \frac{\partial \underline{B}^{(3)*}}{\partial t} &= -ig \left(c A_0^{(2)} \underline{B}^{(1)} - c A_0^{(1)} \underline{B}^{(2)} + \underline{A}^{(1)} \times \underline{E}^{(2)} - \underline{A}^{(2)} \times \underline{E}^{(1)} \right)
\end{aligned}} \quad - (9)$$

The $O(3)$ Gauss and Faraday Laws given the same as those

3) given by Barrett in Table 2, p. 299 of Barrett and Grimes, "Advanced Electromagnetism", in SU(2) notation.

Reduction of Eqn. (9)

We use the fact that $\underline{E}^{(3)} = \underline{0}$ $\boxed{\text{empirically}}$

and that the 4 momentum is:

$$p^\mu = e A^{\mu(3)} \quad \text{— (10)}$$

so:

$$\left.\begin{aligned} A^{\mu(1)} &= (0, \underline{A}^{(1)}) \\ A^{\mu(2)} &= (0, \underline{A}^{(2)}) \\ A^{\mu(3)} &= (A_0^{(3)}, \underline{A}^{(3)}) \\ |\underline{A}^{(3)}| &= A_0^{(3)} \end{aligned}\right\} \text{— (11)}$$

These conditions ~~lead to~~ are combined with:

$$\left.\begin{aligned} c A_0^{(3)} \underline{B}^{(1)} &= \underline{A}^{(3)} \times \underline{E}^{(1)} \\ \underline{A}^{(3)} &= A_0^{(3)} \underline{k} \end{aligned}\right] \text{— (12)}$$

and:

$$\left.\begin{aligned} c A_0^{(3)} \underline{B}^{(2)} &= \underline{A}^{(3)} \times \underline{E}^{(2)} \\ \underline{A}^{(3)} &= A_0^{(3)} \underline{k} \end{aligned}\right\} \text{— (13)}.$$

Eqn. (12) is:

$$c\underline{B}^{(1)} = \underline{k} \times \underline{E}^{(1)} \quad \text{— (14)}$$

and eqn (13) is:

$$c\underline{B}^{(2)} = \underline{k} \times \underline{E}^{(2)}. \quad \text{— (15)}$$

Eqns. (14) and (15) are consistent with the

$$\underline{E}^{(1)} = \underline{E}^{(2)*} = \frac{E^{(0)}}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{-i(\omega t - \kappa z)} \quad — (16)$$

$$\underline{B}^{(1)} = \underline{B}^{(2)*} = \frac{B^{(0)}}{\sqrt{2}} (i\underline{i} + \underline{j}) e^{-i(\omega t - \kappa z)} \quad — (17)$$

$$\underline{A}^{(1)} = \underline{A}^{(2)*} = \frac{A^{(0)}}{\sqrt{2}} (i\underline{i} + \underline{j}) e^{-i(\omega t - \kappa z)} \quad — (18)$$

i.e :

$$\left.\begin{aligned} \underline{E}^{(1)} &= -ic\underline{B}^{(1)} \\ \underline{E}^{(2)} &= ic\underline{B}^{(2)}. \end{aligned}\right] \quad — (19)$$

Now multiply both sides of eqn. (14) by $B^{(0)}$:

$$B^{(0)} \underline{k} \times \underline{E}^{(1)} = cB^{(0)}\underline{B}^{(1)}. \quad — (20)$$

i.e. $\underline{B}^{(3)} \times \underline{E}^{(1)} = cB^{(0)}\underline{B}^{(1)}$, and using eqn. (19):

$$-i\underline{B}^{(3)} \times \underline{B}^{(1)} = B^{(0)}\underline{B}^{(1)}$$

i.e $\boxed{\underline{B}^{(3)} \times \underline{B}^{(1)} = iB^{(0)}\underline{B}^{(2)*}}$ — (21)

Similarly, eqn (15) is:

$$\boxed{\underline{B}^{(2)} \times \underline{B}^{(3)} = iB^{(0)}\underline{B}^{(1)*}} \quad — (22)$$

<u>These are</u> <u>two components</u> of ~~the~~ <u>B cyclic theorem</u>.

Now multiply eqn. (22) by $\underline{B}^{(1)} \times$:

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$$\underline{B}^{(1)} \times (\underline{B}^{(2)} \times \underline{B}^{(3)}) = i B^{(0)} \underline{B}^{(1)} \times \underline{B}^{(2)}$$

$$= \underline{B}^{(2)} (\underline{B}^{(1)} \cdot \underline{B}^{(3)}) - \underline{B}^{(3)} (\underline{B}^{(2)} \cdot \underline{B}^{(1)})$$

$$= - B^{(0)2} \underline{B}^{(3)}$$

$$= - B^{(0)2} \underline{B}^{(3)*}$$

i.e. $$\boxed{\underline{B}^{(1)} \times \underline{B}^{(2)} = i B^{(0)} \underline{B}^{(3)*}} \quad - (23)$$

which is the third component of the B cyclic theorem.

We also have:

$$\underline{A}^{(1)} \times \underline{E}^{(2)} = \underline{A}^{(2)} \times \underline{E}^{(1)}. \quad - (24)$$

Therefore, eqn. (9) is reduced to:

$$\boxed{\begin{aligned} \underline{\nabla} \times \underline{E}^{(1)*} + \frac{\partial \underline{B}^{(1)*}}{\partial t} &= \underline{0} \\ \underline{\nabla} \times \underline{E}^{(2)*} + \frac{\partial \underline{B}^{(2)*}}{\partial t} &= \underline{0} \\ \frac{\partial \underline{B}^{(3)*}}{\partial t} &= \underline{0} \end{aligned}} \quad \longrightarrow (25)$$

Eqn. (5) is reduced to:

$$\boxed{\begin{aligned} \underline{\nabla} \cdot \underline{B}^{(1)*} &= 0 \\ \underline{\nabla} \cdot \underline{B}^{(2)*} &= 0 \\ \underline{\nabla} \cdot \underline{B}^{(3)*} &= 0 \end{aligned}} \quad \longrightarrow (26)$$

EMPIRICAL EVIDENCE FOR EQNS (25) AND (26)

1.) The first two components of eqns. (25) are ~~the~~ Faraday induction laws for the plane waves $\underline{E}^{(1)} = \underline{E}^{(2)*}$ and $\underline{B}^{(1)} = \underline{B}^{(2)*}$.

2.) The third component of eqn (25) is the law for $\underline{B}^{(3)}$, which does **not** give an ~~Faraday~~ $\underline{E}^{(3)}$ by Faraday induction empirically (Raja et al., 1996; Compton et al. at Oak Ridge).

3.) Eqn. (26) is consistent with a solenoidal $\underline{B}^{(1)}$, $\underline{B}^{(2)}$ and $\underline{B}^{(3)}$. There seems to be no empirical evidence for a magnetic monopole of the simple kind.

4.) In $SU(2) \times SU(2)$ ~~theory~~ of the electroweak field, Crowell has shown that $\underline{A}^{(3)}$ is detectable empirically with a heavy hadron collider in high energy particle physics.

5.) There is **no** empirical evidence for a radiated $\underline{E}^{(3)}$ field.

6.) $p^{\mu} = e A^{\mu(3)}$ gives a classical explanation of the Compton effect for radiation reaction through:

$$\boxed{eA^{\mu(3)} = \hbar \kappa^{\mu}}$$

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COMMENTS, EQN. (9).

1) The terms $cA_0^{(2)}\underline{B}^{(3)}$; $cA_0^{(1)}\underline{B}^{(3)}$ and $cA_0^{(1)}\underline{B}^{(1)} - cA_0^{(1)}\underline{B}^{(2)}$ are not compatible in direction with the l.h.s. Therefore:

$$\boxed{A_0^{(1)} = A_0^{(2)} = 0}$$

since $\underline{B}^{(1)}$, $\underline{B}^{(2)}$ and $\underline{B}^{(3)}$ are non-zero.

2) The other terms are compatible in direction.

3) The above suggests the radiation gauge for the transverse modes:

$$\boxed{\begin{aligned} \underline{\nabla} \cdot \underline{A}^{(1)} &= \underline{\nabla} \cdot \underline{A}^{(2)} = 0 \\ A_0^{(1)} &= A_0^{(2)} = 0 \end{aligned}}$$

4) Additionally: $\boxed{A^{\mu(3)} = (A_0^{(3)}, \underline{A}^{(3)})}$

which quantizes to photon four-momentum:

$$\boxed{eA^{\mu(3)} = \hbar \kappa^{\mu}}$$

so the scalar potential, $A_0^{(3)}$, is longitudinal.

5) The Lorentz condition applies to the longitudinal $A^{\mu(3)}$:

$$\boxed{\partial_\mu A^{\mu(3)} = 0}$$

but it does not apply to the transverse modes.

6) $\boxed{\begin{aligned} &\text{Energy conservation is:} \\ &\qquad \frac{d}{dt} A^{\mu(3)} = 0 \\ &\text{Planck energy} = eA_0^{(3)}/c = \hbar\omega \\ &\text{Planck momentum} = e|\underline{A}^{(3)}| = \hbar\kappa. \end{aligned}}$