SUBSIDIARY CONDITION FOR R

Begin with the Evans Lemma:

$$D^\mu \left(D_\mu q^a_\nu \right) := 0 \qquad -(1)$$

i.e.

$$\partial^\mu \left(D_\mu q^a_\nu \right) := 0 \qquad -(2)$$

or

$$\partial^\mu \left(\partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda - \Gamma^\nu_{\mu\lambda} q^a_\nu \right) := 0 \qquad -(3)$$

$$\Rightarrow \qquad \boxed{\Box q^a_\mu = R q^a_\mu} \qquad -(4)$$

where:

$$R = q^\lambda_a \partial^\mu \left(\Gamma^\nu_{\mu\lambda} q^a_\nu - \omega^a_{\mu b} q^b_\lambda \right). \qquad -(5)$$

Eqn (4) is an eigen equation for R and eqn. (5) is a subsidiary condition for R. In general they must be solved simultaneously. In these notes R is re-expressed in terms only of the connections, the advantage of this is that R in eqn. (4) is an eigenvalue of the tetrad, and R in eqn (5) is a scalar curvature defined

2) similarly to its well known definition in the 1915 theory as a contraction of the Riemann tensor via the Ricci tensor. Thus eqn (4) is an expression for R in quantum mechanics, and eqn. (5) is an expression for R in classical field theory.

Before embarking on this proof we note that the tetrad postulate is the same as the first Cartan structure equation. The tetrad postulate is:

$$\partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda = \Gamma^\nu_{\mu\lambda} q^a_\nu \quad -(6)$$

Now use the result:

$$\partial_\lambda q^a_\mu + \omega^a_{\lambda b} q^b_\mu = \Gamma^\nu_{\lambda\mu} q^a_\nu \quad -(7)$$

and

$$T^a_{\mu\lambda} = T^\nu_{\mu\lambda} q^a_\nu \quad -(8)$$

where the torsion tensor is:

$$T^\nu_{\mu\lambda} = \Gamma^\nu_{\mu\lambda} - \Gamma^\nu_{\lambda\mu}. \quad -(9)$$

Subtracting (7) from (6) we find <u>the first Cartan structure equation</u>:

$$d \wedge q^a_{\ \lambda} + \omega^a_{\ \mu b} \wedge q^b_{\ \lambda} = T^a_{\ \mu\lambda} \qquad -(10)$$

or

$$D \wedge q^a = T^a . \qquad -(10a)$$

Therefore the first Cartan structure equation is the difference of two tetrad postulates.

Now proceed to develop eqn. (5) using:

$$\partial^\mu = g^{\mu\sigma} \partial_\sigma \qquad -(11)$$

to give:

$$R = g^{\mu\sigma} q^\lambda_{\ a} \left(\Gamma^\nu_{\mu\lambda} \partial_\sigma q^a_{\ \nu} + q^a_{\ \nu} \partial_\sigma \Gamma^\nu_{\mu\lambda} - \omega^a_{\ \mu b} \partial_\sigma q^b_{\ \lambda} - q^b_{\ \lambda} \partial_\sigma \omega^a_{\ \mu b} \right) . \qquad -(12)$$

Now use the tetrad postulate again:

$$\partial_\sigma q^a_{\ \nu} + \omega^a_{\ \sigma b} q^b_{\ \nu} - \Gamma^\rho_{\sigma\nu} q^a_{\ \rho} = 0 \qquad -(13)$$

$$\partial_\sigma q^b_{\ \lambda} + \omega^b_{\ \sigma c} q^c_{\ \lambda} - \Gamma^\nu_{\sigma\lambda} q^b_{\ \nu} = 0 \qquad -(14)$$

to find:

$$R = g^{\mu\sigma}\Big(q^{\lambda}_{a} q^{a}_{\rho} \Gamma^{\nu}_{\mu\lambda} \Gamma^{\rho}_{\sigma\nu} - q^{\lambda}_{a} q^{b}_{\nu} \Gamma^{\nu}_{\mu\lambda} \omega^{a}_{\sigma b} + q^{\lambda}_{a} q^{a}_{\nu} \partial_{\sigma} \Gamma^{\nu}_{\mu\lambda} - q^{\lambda}_{a} q^{b}_{\nu} \omega^{a}_{\mu b} \Gamma^{\nu}_{\sigma\lambda} + q^{\lambda}_{a} q^{c}_{\lambda} \omega^{a}_{\mu b} \omega^{b}_{\sigma c} - q^{\lambda}_{a} q^{b}_{\lambda} \partial_{\sigma} \omega^{a}_{\mu b} \Big) \quad -(15)$$

Now eliminate the tetrads using:

$$R^{\sigma}{}_{\lambda\nu\mu} = q^{\sigma}_{a} q^{b}_{\lambda} R^{a}{}_{b\nu\mu} \quad -(16)$$

$$R^{a}{}_{b\nu\mu} = \partial_{\nu} \omega^{a}_{\mu b} - \partial_{\mu} \omega^{a}_{\nu b} + \omega^{a}_{\nu c} \omega^{c}_{\mu b} - \omega^{a}_{\mu c} \omega^{c}_{\nu b} \quad -(17)$$

$$R^{\sigma}{}_{\lambda\nu\mu} = \partial_{\nu} \Gamma^{\sigma}_{\mu\lambda} - \partial_{\mu} \Gamma^{\sigma}_{\nu\lambda} + \Gamma^{\sigma}_{\nu\rho} \Gamma^{\rho}_{\mu\lambda} - \Gamma^{\sigma}_{\mu\rho} \Gamma^{\rho}_{\nu\lambda} \quad -(18)$$

$$R^{a}{}_{b\nu\mu} = q^{a}_{\sigma} q^{\lambda}_{b} R^{\sigma}{}_{\lambda\nu\mu} \quad -(19)$$

The Riemann tensor and the Riemann form are antisymmetric respectively in σ and λ and in a and b. Using this antisymmetry eqn. (15) reduces to:

$$R = -g^{\mu\sigma} q^{\lambda}_{a} q^{b}_{\nu} \left(\Gamma^{\nu}_{\mu\lambda} \omega^{a}_{\sigma b} + \omega^{a}_{\mu b} \Gamma^{\nu}_{\sigma\lambda} \right) \quad -(20)$$

5) Now simplify and remove the tetrads using:

$$q_a^{\lambda}\Gamma^{\nu}_{\mu\lambda} = \Gamma^{\nu}_{\mu a} \quad -(21)$$

$$q^b_{\nu}\omega^a_{\sigma b} = \omega^a_{\sigma\nu} \quad -(22)$$

$$q^{\lambda}_a\omega^a_{\mu b} = \omega^{\lambda}_{\mu b} \quad -(23)$$

$$q^b_{\nu}\Gamma^{\nu}_{\sigma\lambda} = \Gamma^b_{\sigma\lambda} \quad -(24)$$

to give:

$$R = -g^{\mu\sigma}\left(\Gamma^{\nu}_{\mu a}\omega^a_{\sigma\nu} + \omega^{\lambda}_{\mu b}\Gamma^b_{\sigma\lambda}\right) \quad -(25)$$

Finally re-arrange dummy indices $b \rightarrow a$; $\lambda \rightarrow \nu$:

$$\boxed{R = -g^{\mu\sigma}\left(\Gamma^{\nu}_{\mu a}\omega^a_{\sigma\nu} + \omega^{\nu}_{\mu a}\Gamma^a_{\sigma\nu}\right)} \quad -(26)$$

Eqn (26) is the desired generalization of ~~the~~ the equation for R in the 1915 theory:

$$R = g^{\mu\sigma}R_{\mu\sigma} \quad -(27)$$

6) Comparing eqns (26) and (27) gives the Ricci tensor in the Evans unified field theory:

$$\boxed{R_{\mu\sigma} = -\left(\Gamma^{\sim}_{\mu a}\,\omega^{a}_{\sigma\sim} + \omega^{\sim}_{\mu a}\,\Gamma^{a}_{\sigma\sim}\right)}$$

— (28)

Using the index-contracted Einstein field eqn:

$$R = -kT \quad -(29)$$

it is found that:

$$\boxed{kT = g^{\mu\sigma}\left(\Gamma^{\sim}_{\mu a}\,\omega^{a}_{\sigma\sim} + \omega^{\sim}_{\mu a}\,\Gamma^{a}_{\sigma\sim}\right)}$$

— (30)

Eqn. (30) is the classical subsidiary condition of the Evans wave equation of quantum mechanics:

$$\boxed{(\Box + kT)\,q^{a}_{\mu} = 0.} \quad -(31)$$