

Special Parameters of a Curve in the Four-Manifold

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Abstract

It is demonstrated that a four-manifold at least of signature $(+,+,+,-)$ is characterized by two special parameters of any curve¹ in it, which reflect the properties of the manifold. That are the curvature vector and a special asymmetric tensor of second rank. They accompany the curve.

Any curve in a n -manifold is accompanied by a special orthogonal enuple or n -bein. That are n mutually orthogonal unit vectors. The vectors of the accompanying n -bein are determined by the generalized FRENET formulae according to BLASCHKE (EISENHART [1]).

The first vector is the tangent vector of the curve

$$t^i = \frac{dx^i}{ds} \quad . \quad (1)$$

The second vector is the main normal

$$n^i = \rho t^a t^i_{;a} \quad , \quad (2)$$

in which ρ means the curvature radius.

The most important curve parameter might be the curvature vector

$$k^i = \frac{n^i}{\rho} = t^a t^i_{;a} = \frac{d^2 x^i}{ds^2} + \{^i_a\} t^a t^b \quad . \quad (3)$$

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¹under consideration of the peculiarities from the signature

Are there other curve parameters ?

The four-manifold opens the possibility of parameters that are performed from the congruences of two dual surfaces instead of single vectors. A parameter analogous the curvature vector were an asymmetric tensor of second rank, and must be written with these surfaces.

This asymmetric tensor is expressible from a vector potential

$$F_{ik} = A_{i,k} - A_{k,i} \quad . \quad (4)$$

If the divergences vanish

$$F^{ia}{}_{;a} = 0 \quad , \quad (5)$$

it exists a relation to the RICCI tensor

$$R_{ik} = \frac{1}{4} g_{ik} F_{ab} F^{ab} - F_{ia} F_k{}^a \quad . \quad (6)$$

(A constant part of the RIEMANNIAN curvatures would not disturb the derivation, but the description were not more clear.)

Equ. (4) to (6) involve a special kind of RIEMANNIAN geometry of the four-manifold of signature $(+, +, +, -)^2$. This geometry leads indeed to special dual surfaces, which perform just mentioned asymmetric tensor. As well, the derivation follows in general that of the RICCI main directions as done by EISENHART [1]. Unlike all other manifolds, the results for the four-manifold involve two main surfaces instead of four main directions.

The RICCI main directions (written in terms according to EISENHART) follow from

$$\det |R_{ik} + \rho g_{ik}| = 0 \quad (7)$$

with the solutions

$$\rho_{|1} = \rho_{|4} = +\rho_o \quad , \quad \rho_{|2} = \rho_{|3} = -\rho_o \quad (8)$$

with

$$\rho_o^2 = R_1{}^a R_a^1 = R_2{}^a R_a^2 = R_3{}^a R_a^3 = R_4{}^a R_a^4 \quad . \quad (9)$$

²It were to investigate if this signature is necessary.

Characteristical are the two double-roots, that means: There are two dual surfaces of the congruences $e_{|1}{}^i e_{|4}{}^k - e_{|1}{}^k e_{|4}{}^i$ and $e_{|2}{}^i e_{|3}{}^k - e_{|2}{}^k e_{|3}{}^i$ with extreme mean RIEMANNIAN curvature. $e_{|1} \dots e_{|4}$ are the vectors of an orthogonal quadrupel (vierbein) in those “main surfaces”.

With it we get

$$g_{ik} = e_{|1-i} e_{|1-k} + e_{|2-i} e_{|2-k} + e_{|3-i} e_{|3-k} - e_{|4-i} e_{|4-k} \quad , \quad (10)$$

$$\frac{R_{ik}}{\rho_o} = -e_{|1-i} e_{|1-k} + e_{|2-i} e_{|2-k} + e_{|3-i} e_{|3-k} + e_{|4-i} e_{|4-k} \quad . \quad (11)$$

If we set

$$c_{|ik} = -c_{|ki} = F_{ab} e_{|i}{}^a e_{|k}{}^b \quad (12)$$

follows

$$-((c_{|23})^2 + (c_{|14})^2) = 2\rho_o \quad , \quad (13)$$

$$c_{|12} = c_{|34} = c_{|13} = c_{|24} = 0 \quad . \quad (14)$$

With it, the asymmetric tensor

$$F_{ik} = -c_{|14}(e_{|1-i} e_{|4-k} - e_{|1-k} e_{|4-i}) + c_{|23}(e_{|2-i} e_{|3-k} - e_{|2-k} e_{|3-i}) \quad (15)$$

is performed from the main surfaces !

That means, this tensor is indeed a curve parameter too. It accompanies the curve like the curvature vector does it.

Both curve parameters, the curvature vector and this tensor, reflect the properties of the four-manifold.

References

- [1] EISENHART, L.P.: Riemannian Geometry, Princeton university press.